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Macroscopic Realizations of Quantum Logics

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INTRODUCTION

If one thinks about quantum mechanics as a realization of some new quantum logics corresponding to non-Boolean lattices then one can ask the question: are there some systems or situations, maybe apart from microphysics, which are realizations of quantum logics? Examples of such realizations are very important because by using them we can simulate microscopic quantum systems by macroscopic ones.

This also can be useful for computation in quantum physics. For example, it is well known that instead of solving some nonlinear equation like $\Delta\varphi = -4\pi \exp(-2\alpha\varphi)$ one can just take an electrolyte and measure in it the electric potential φ in different points. In such a way a mathematical calculation can be simulated by a physical experiment.

In this sense we can contemplate a "quantum computer" being a macroscopic device working according to "quantum logic" corresponding to non-Boolean lattices of properties of a quantum microparticle (proton, quark, etc). This device could give answers on many questions in particle physics. The other practical use of such a "quantum computer" is, as discussed in Deutch (1985), the possibility of a new kind of calculation due to the existence of noncommuting observables. For example, using measurements of coordinates and momenta, one could very rapidly perform Fourier transform of a function.

In this paper we give examples of such macroscopic realizations for simple cases simulating one- and two-particle systems with spin 1/2.

D. Finkelstein was the first to see the correspondence between quantum lattices and graphs which makes it possible to find macroscopic realizations of quantum logics. To illustrate the idea, consider a very simple quantum

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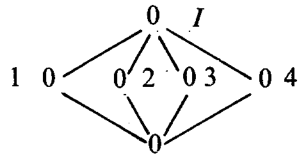
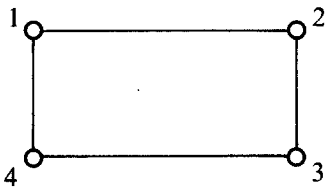
system: a particle with spin one-half which is described by two projections of spin S_z and S_x. The lattice of properties of this particle is an orthomodular nondistributive lattice:



To this lattice there corresponds the graph (Finkelstein and Finkelstein, 1982)



Now consider the opposite question: in what sense does the nondistributive lattice correspond to the graph? Let 1, 2, 3, 4 be states of some system (e.g., an economic one) and suppose that there is an observer who tries to check the state of the system by putting questions to it. The system has the following property: it can answer "yes" to the question "are you in 2?" not only if it is in 2, but also if it is in 1 or 3. It can change its state by one step responding to the question if and only if corresponding states are connected by an arc. But let the observer be clever enough to know this property of the system: then he or she must use some "negative logics," and concludes that the system is in 2 if to the question "are you in 4?" a negative answer is obtained. So by a negative answer to a complementary question the observer can know the real state of system. But then it is easy to see that one can find no such questions the negative answer to which corresponds to the state "1 or 2," "2 or 3," and so on. This means that in our "negative logics," "1 or 2" coincides with I, "any state." One cannot find any difference between disjunctions $1 \vee 2$, $2 \vee 3$, $3 \vee 4$, $1 \vee 4$, and I. That is why the lattice is nondistributive, $1 \vee 2$ is true *if* 1 is true, $1 \vee 2$ is true if 2 is true, but not *only if*: $1 \vee 2$ can be true when both 1 and 2 are false.

It is easy to see that our observer cannot use classical probability theory for the system he or she controls, because there is no probability measure for a nondistributive lattice. For example, in the symmetrical case, the probability of each state must be 1/4. But since $1 \vee 2 = I$, the probability of $1 \vee 2$ must be equal to 1, but it is equal to $1/4 + 1/4 = 1/2$. In the following sections we give some rigorous results for constructing macroscopic realizations of quantum logics.

The possibility of simulating quantum systems by classical automata described by graphs is important for two reasons. The first one is that it gives the way to construct quantum computers—automata composed of classical elements but working analogously to quantum systems due to quantum logics. The second reason is that it has the profound meaning of showing why we can speak about quantum objects in terms of classical experiment. In some sense it corresponds to Lüdwig's (1989) extreme point of view that "atoms do not exist" and only classical measuring apparata really "exist," and "quantum objects" are merely a language describing relations between classical bodies and logics of these relations.

Nevertheless we consider Ludwig's interpretation extreme because classical bodies "consist" of quantum objects, but not the opposite, and macroscopic quantum mechanics gives us reason to believe that there are no purely classical objects.

After constructing graphs and property lattices for one- and two-particle systems we give the rule for defining the wave function in terms of weights on graphs. Then we show how Bell's inequalities can be violated on graphs. This yields an example of Bell's inequalities breaking for classical systems with non-Boolean logics.

At the end of the paper we discuss the problem of wave packet reduction for two-particle system, which is connected with the Boolean nature of consciousness. Properties described by non-Boolean lattices do not correspond to events in Minkowski space-time and there is no usual probability attached to them. It is only due to the "Booleization" of the lattice by an observer that they become events. This Booleization is done by means of time, namely the observer can check values of noncommuting observables by measuring them at different moments of time. So, the observer must move in time in order to apprehend through her or his Boolean consciousness the non-Boolean properties of quantum system. We think this can be an explanation of why we all move in time.

1. GRAPHS ASSOCIATED WITH PHYSICAL SYSTEMS

Consider an experimental plant with two kinds of controls: the source and the analyzer. It is assumed that the source prepares the plant, and the analyzer can be tuned to verify some finite set of properties. The result of each verification is the answer: either YES or NO. The procedure of verification is also called putting the question. So, the observer can:

1. Change the parameters of preparation by tuning the knobs of the source.

2. Put different questions by pressing the *buttons* (controls of the analyzer) and obtain *answers*.

Now let us enumerate the control buttons of the device by 1, 2, . . . , N. Let n_i be the total number of checks of the question i (pressing the button i, $i=1, \ldots, N$), and let m_i be the number of answers YES to the question i. If under multiple trials the stable frequencies appear

$$p_i = m_i/n_i$$

for each question $i=1, \ldots, N$, we way that the setted positions of the controls of the source provide the preparation of a certain state. The *state* will be identified with the set of frequencies p_i. Since the trials are independent, we are not able to discern positions of the knobs of the source but yielding the same sets of stable frequencies. Therefore the set of all states *observed in the described experiment* is identified with the set of all collections p_i we can observe in this experiment. These collections evidently form a subset θ of an N-dimensional cube:

$$\theta \subset [0, 1]^n$$

The set of states is convex, since we can always insert a classical roulette to our plant which presses buttons j, k with any prior probabilities λ, μ, $\lambda+\mu=1$.

Consider two questions j, k. Consider the set θ_j of all states of the object for which there is also the answer YES to the question j. If for any state from θ_j the answer to the question k is always NO, then the question j is called *excluding* the question k; denote it by *jlk*.

If the questions j, k do exclude each other, they are called mutually exclusive, or orthogonal:

$$j \perp k \overset{\text{def}}{\Leftrightarrow} j \llcorner k \;\&\; k \llcorner j$$

The relation $\perp$ on the set N of all buttons (questions) is symmetric. Consider the relation P, which is the complement (in the set-theoretic sense) of

$$jPk \Leftrightarrow j \neq k$$

Following Finkelstein and Finkelstein (1982), we call the graph of the relation i the *O-graph* of the object, and the graph of P is called the *P-graph* of the object. Both are nonoriented.

We consider P-graphs further. We shall call them graphs of objects. Briefly:

The graph of the object is the graph G whose vertices 1, . . . , N are associated with questions, and edges connect nonorthogonal vertices, and only them.

The result of any individual experiment can be considered as the outcome of a trial. In accordance with the conventional theoreticoprobabilistic approach a property of the object is a subset of the set of elementary outcomes. In the described situation the space is the space of all questions, or, in other words, the set of all vertices of the graph of the object. Consider in detail which subsets of this set can be called properties.

Let A be a property and let the object be emitted by the source in a state $W=\{p_i\}$. The first way to define a property is to say: the object possesses A iff the answer YES is obtained with probability 1 for any question contained in A. However, this definition is not operationalistic. Indeed, in each experiment we put only one question from the collection A. Yet this does not assure us that to any other question from A a positive answer is obtained, too.

The second way we propose to define a property is to convince oneself that the object does *not* possess the property A. That means that for each question orthogonal to each question from A the positive answer is always obtained. This set of questions will be called the negation of A, or *NOT A*:

$$\bar{A} = \text{NOT } A = \{j \mid \forall k \in A, j \perp k\}$$

A property of the object is defined as a subset of the set V of vertices for which the double negation rule is valid. Denote the set of all properties of the object by L:

$$L = \{A \subset V \mid A = \bar{\bar{A}}\}$$

The set L is an ortholattice possessing three operations:

1. Negation $A \mapsto A^{\perp}$. The operation is an involution in accordance with the definition given above:

$$A^{\perp} = \bar{A}$$

2. Meets (conjunctions) $A \wedge B$ are defined as set-theoretic intersections:

$$A \wedge B = A \cap C$$

3. Joins (disjunctions) $A \vee B$ are defined following De Morgan's law:

$$A \vee B = (A^{\perp} \wedge B^{\perp})^{\perp} = \overline{(\bar{A} \cap \bar{B})}$$

These operations really form an ortholattice structure on the set L since this is a special case of the Birkhoff–Ore polarity construction.

2. FROM GRAPHS TO LATTICES AND BACK

The algebraic feature of the property lattices of graphs allow us to formulate an unambiguous algorithm for constructing the property lattice by a given graph G. This algorithm consists of two stages. At the first step the maximal proper (not coinciding with the greatest element of L) elements are built. At the second step all their possible intersections are built. Let us consider this in detail.

Given a nonoriented graph G, denote by V the set of its vertices, and by E the set of its edges.

Step A. For each vertex $j \in V$ consider the set $\{j\}^{\perp}$ of all vertices not connected with j,

$$\{j\}^{\perp} = \{k \in V/(k, j) \notin E\}$$

Step B. Form all possible intersections of $\{j\}^{\perp}$'s, placing them into the Hasse diagram.

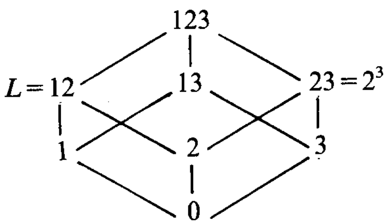
We consider several examples. For brevity we omit the brackets denoting collections: e.g., 23 means $\{2, 3\}$ and so on.

Example 1. The totally disjoint graph of these vertices:

$$G = 1^* \, 2^* \, 3^*, \qquad V = \{1, 2, 3\}, \qquad E = \varnothing$$

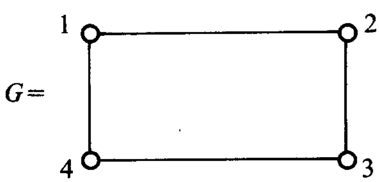
Step A. $1^{\perp} = 23$, $2^{\perp} = 13$, $3^{\perp} = 12$.

Step B. $23 \cap 13 = 3$, $23 \cap 12 = 2$, $13 \cap 12 = 1$. Place the obtained six elements 23, 13, 12, 3, 2, 1 into the Hasse diagram, adding the greatest $(I = V = 123)$ and the least (0) elements of L:

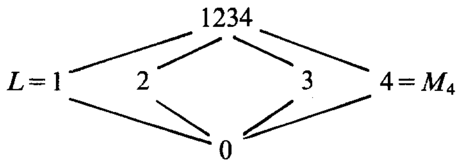


We see that L is the Boolean lattice 2^3. Moreover, for any N-vertex totally disjoint graph the property lattice is always the Boolean lattice with N atoms.

Example 2. This graph will be used in simulating Bell's inequalities (see Section 6).

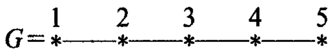


Step A. $1^{\perp}=3$, $2^{\perp}=4$, $3^{\perp}=1$, $4^{\perp}=2$.
Step B. No more elements, although



In this case L is a nondistributive modular ortholattice M_4 with four atoms.

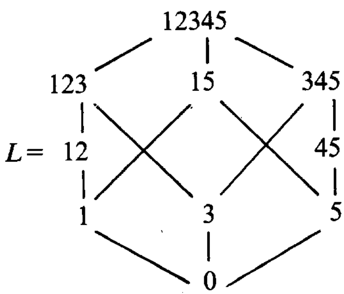
Example 3. This graph yields a nonmodular ortholattice:



Step A. $1^{\perp}=345$, $2^{\perp}=45$, $3^{\perp}=15$, $4^{\perp}=12$, $5^{\perp}=123$.
Step B. $345 \cap 15=45 \cap 15=5$, $345 \cap 123=3$, and

$$15 \cap 12=15 \cap 123=1$$

We have



One can see that L is neither distributive nor even modular.

As already mentioned, given a finite ortholattice L, one can always construct a graph G whose property lattice is L. Consider the algorithm for the reconstruction of a graph by its property lattice.

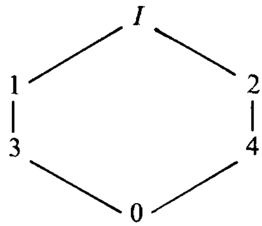
Given a finite ortholattice L with orthocomplements denoted by $\perp$, an element $j \in L$ is called join-irreducible if it cannot be represented in L as a join of elements different from j.

Step 1. The set V of vertices of a future graph G is the set of all nonzero join-irreducible elements of L:

$$V=\{j \in L \mid \nexists k, l/k \neq j, l \neq j, k \vee l=j\}$$

Step 2. Examples of the edges of G connecting nonorthogonal elements of lattices are (i) 1, 2, 3; (ii) 1, 2, 3, 4; (iii) 1, 2, 3, 5, 12, 45.

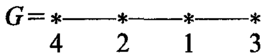
Example 4. Consider



$1^{\perp}=4, \quad 2^{\perp}=3$

Step A. Join-irreducibles are 1, 2, 3, 4.

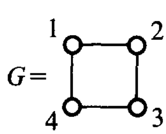
Step B. 1 is orthogonal only to 4, thus it must be connected with 2 and 3. 2 is orthogonal to 3 only, so it is joined to 3 only, so it is joined with 1 and 4. 3 is orthogonal to 2 and 4 and is connected only with 1. Analogously, 4 is connected only with 2. The graph G is



We emphasize that given an arbitrary graph G, we can construct its property lattice L and then reconstruct the graph in accordance with the algorithm described above, through we get in general a subgraph of G. For detailed mathematical treatment of this question see Zapatrin (1990, n.d.).

3. LOGICAL DESCRIPTION OF SPIN-1/2 PARTICLE

Grib and Zapatrin (1990) describe spin-1/2 particles in terms of graphs endowed with probabilistic weights. For our purposes we shall need a more special construction, namely, we restrict spin measurements to the situation of the graph G of Example 2,



whose vertices are associated with the following questions:

1. "$S_z=+1/2$?"
2. "$S_x=+1/2$?"
3. "$S_z=-1/2$?"
4. "$S_x=-1/2$?"

to which correspond the following vectors in the state space $H = \mathbb{C}^2$ of the particle:

$$1: \quad \langle Z_+| = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$2: \quad \langle X_+| = e_2 = 2^{-1/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$3: \quad \langle Z_-| = e_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$4: \quad \langle X_-| = e_4 = 2^{-1/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

As mentioned in Section 1, an observable state of the object is associated with the collection of weights $p_1, \ldots, p_4$ assigned to the vertices of G. These weights are not arbitrary; evidently for any state $\{p_i\}$

$$0 \le p_i \le 1, \qquad i = 1, 2, 3, 4$$

$$p_1 + p_3 = p_2 + p_4 = 1$$

A pure state in the experiment in question is the state corresponding to the definite value $+1$ of the spin projection on an axis in the XZ plane forming an angle γ $(0 \le \gamma \le 2\pi)$ with the Z axis. In $\mathbb{C}^2$ this state is described by the vector

$$\langle p| = \begin{pmatrix} \cos \gamma/2 \\ \sin \gamma/2 \end{pmatrix}$$

Therefore, in accordance with the traditional quantum rules the weights, p_i are

$$P_i = |\langle p|e_i\rangle|^2, \qquad i = 1, 2, 3, 4$$

$$p_1 = \cos^2 \gamma/2, \qquad p_2 = (1 + \sin \gamma)/2$$

$$p_3 = \sin^2 \gamma/2, \qquad p_4 = (1 - \sin \gamma)/2$$

For two states $\langle p|$ and $\langle q|$ the transition probability is equal to $|\langle p|q\rangle|^2$. If

$$\langle q| = \begin{pmatrix} \cos \delta/2 \\ \sin \delta/2 \end{pmatrix}$$

Then

$$P_{pq}=|\langle p|q\rangle|^2=\cos^2(\gamma-\delta)/2$$

In terms of weights on graphs the transition probability can be directly calculated and has the form

$$p_{pq}=\sum_{i=1}^{4} p_i q_i - 1/2$$

or, equivalently,

$$P_{pq}=p_i T_i^k q_k + K$$

where $K=0$,

$$T_i^k=\begin{cases}1, & \text{if } i=k\\ 0, & \text{if } i\perp k\\ -1/4 & \text{otherwise}\end{cases}$$

and the summation by repeated indices is performed over all vertices of G.

4. COUPLED SYSTEMS

First we briefly recall what is meant by the graph description of a quantum system S associated with Hilbert space H. We select some properties (closed subspaces of H) and associate with them vertices of a graph G. The edges of the graph G connect only the vertices associated with nonorthogonal subspaces. The obtained graph G is called the graph of the system S.

Now consider two quantum systems S_1 and S_2 associated with Hilbert spaces H_1 and H_2 and graphs G_1 and G_2, respectively. The Hilbert space of the compound system S is the tensor product $H=H\otimes H$. Building the graphs G_1 and G_2, we select some subspaces in H_1 and H_2. All pairwise products of those subspaces generated a collection of subspaces of H. To each subspace of this collection we associate a vertex of the product graph G. Thus, the set of vertices of G is the set of all ordered pairs of vertices of G_1 and G_2. The orthogonality on the set of such pairs is inherited from the lattice

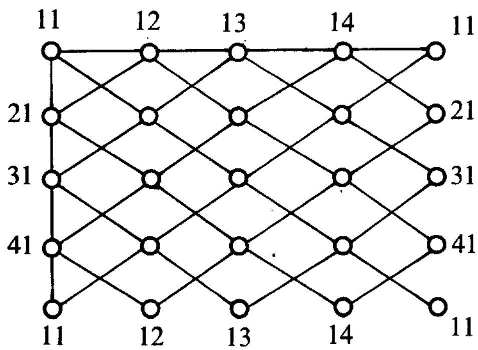
$L(H)$, namely for i, $i' \in L(H_1)$ and k, $k' \in L(H_2)$,

$$(i, k) \perp i', k') \Leftrightarrow i \perp k \quad \text{or} \quad i' \perp k'$$

Thus, the edges of the product graph connect two vertices (i, k) and (i', k') if both pairs of vertices i, i' of G_1 and k, k' of G_2 are connected by the edges of G_1 and G_2, respectively.

Now consider two spin-1/2 particles and restrict possible spin measurements on the XZ plane. This situation is described in Section 3.

The product graph $G = G_1 \times G_2$ has $4 \times 4 = 16$ vertices of the form ik, where $i = 1, 2, 3, 4$. In accordance with the above definition, the graph G has the following form:



This is the planar development of the product graph G (vertices labeled by the same indices are identical).

When a system S is represented by its graph G the (experimentally distinguishable) states of S are described by endowing the vertices of G with probability weights (interpreted as the possibility of occurrence of the corresponding property). Let S be initially in a state $A = \{a_i\}$ described by a collection $\{a_i\}$ of probability weights on the vertices of G. Then the probability of finding S in a state $B = \{b_p\}$ is calculated by the transition probability formula:

$$P_{AB} = a_i T_i b_p + k$$

where summation by repeated indices is performed over all vertices of G. Here T_i^p is a symmetric matrix and K is a constant, both depending only on the form of the graph G. Denote the set of all states on G by $\Phi(G)$.

Here we consider the graph G whose vertices are labeled by double indices. Since G is a product graph we can consider two kinds of states.

Factorizable states are represented as pairwise products of weights on G_1 and G_2. In other words, for $\{c_{ik}\} \in \Phi(G_1 \otimes G_2)$

$$(\{c_{ik}\} \text{ is factorizable}) \overset{\text{def}}{\Leftrightarrow} \exists \{a_i\} \in \Phi(G_1),\ \exists \{b_i\} \in \Phi(G_2) | c_{ik} = a_i b_k$$

If this condition does not hold for a state c_{ik}, it is called nonfactorizable.

The transition probability formula for the graph G has the form

$$P_{CD} = C_{ik} T_{ik}^{pq} d_{pq} + K$$

where $K = 5/4$, and

$$T_{ik} = \begin{cases} 1 & \text{if } p = i \text{ and } q = k \\ 0 & \text{if } p \perp i \text{ and } q \perp k \\ -1/4 & \text{otherwise} \end{cases}$$

[for graphs G_1, G_2 from Section 3, $p \perp i$ means $p - i = 2 \pmod 4$].

The values of T_{ik}^{pq} and K can be obtained from the requirement of correspondence with traditional quantum mechanical results. In general, given the matrices T_i^p and T_k^q and the constants K_1 and K_2 for the graphs G_1 and G_2, one can obtain the matrix T_{ik}^{pq} and the constant K for the product graph $G = G_1 \otimes G_2$ from: (a) the requirement that both T_{ik}^{pq} and K do not depend on the values of weights (i.e., they are really constants), and (b) the assumption that the two systems S_1 and S_2 are independent and thus for any pair of factorizable $c_{ik} = a_i b_k$ and $d_{pq} = t_i s_q$ the transition probability is the product

$$P_{CD} = P_{AT} P_{BS}$$

where $C = \{c_{ik}\}, \ldots, S = \{s_q\}$.

Now construct the property lattice $L(G)$ corresponding to the graph G. The maximum element I of $L(G)$ is the 16-element set $V(G)$ of all vertices of the graph. The upper row has 16 elements of the form $\{i*, *k\}$, where * runs over 1, 2, 3, 4. Each element of the upper row of $L(G)$ is a 7-element subset of the set $V(G)$ of all vertices of G. The next row downward consists of elements of three kinds: $\{i*\}$ and $\{*k\}$, which are 4-element subsets of $V(G)$, and $\{ik, lm\}$ $(i \neq l, k \neq m)$, the 2-element subsets. The direct calculation shows that there are 4 elements of the form $\{i*\}$ and $\{*k\}$ and 72 elements of the form ik, lm $(i \neq 1, k \neq m)$. The next row is the lowest. It consists of 16 elements which are one-element subsets ik corresponding to each vertex $ik \in V(G)$. At the bottom of $L(G)$ is the void set $\varnothing$. Thus, we have completely described the property lattice $L(G)$ as the lattice of subsets of $V(G)$ partially ordered by set-theoretic inclusion.

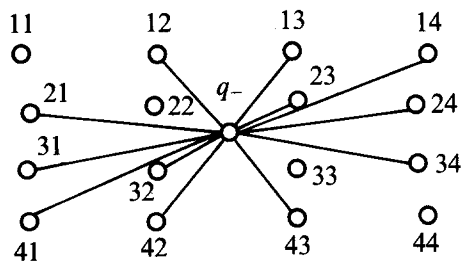
The Hasse diagram of the lattice $L(G)$ consisting of $1+16+80+16+1=114$ elements is too complicated for typographical representation. However, its representation by means of graphs is unambiguous. The mathematical treatise of questions concerning graph representations of ortholattices is given in Zapatrin (1990, n.d.).

5. NONLOCAL QUESTIONS

The permutation operator has two eigenvalues: ± 1. To -1 corresponds the vector $\langle q_-|$ in H: $\langle q_-| = 1/\sqrt{2}\,(e_{12}-e_{21})$, and to $+1$ corresponds its orthogonal complement.

First we introduce into the graph G a new vertex corresponding to the following question for the system: is the wave function antisymmetric? The vertex q associated to this question is connected with other vertices of G according to the rule described in Section 2: two vertices are not connected by an arc if the subspaces associated to them are orthogonal, otherwise we draw an arc. Calculating directly the scalar products $\langle q_-|ik\rangle$, we obtain that $\langle q_-|$ is orthogonal only to vertices 11, 22, 33, and 44. Thus, the vertex q_- must be connected with all vertices of G except the above mentioned. We can also introduce the vertex q_+ associated with the subspace corresponding to $+1$. In this case we should connect q_+ with all vertices except q_-. However, this vertex q_+ will be redundant, namely the property lattice of this graph will be isomorphic to that of the graph H.

The graph H obtained from G by adding the vertex q_- is



Only additional edges are shown; edges of G are omitted.

The property lattice $L(H)$ can be constructed from $L(G)$ described in Section 2 by adding two new elements. The first of them $\{q_-\}$ is an atom (placed in the lowest row together with $\{ik\}$'s). The other one $\{11, 22, 33, 44\}$ is placed in the upper row. Also the vertex q_- must be added to 2-element subsets of the form $\{ik, ki\}$ of the middle row. This completes the description of the lattice $L(H)$.

6. BELL'S INEQUALITIES AND THEIR BREAKING IN TERMS OF GRAPHS

Let X, Y, Z be some elementary questions, and $\bar{X}$, $\bar{Y}$, $\bar{Z}$ be their negations considered for a graph G. An elementary question means checking being in the state described by the collection of vertex weights $\{a_q\}$, where q runs over all vertices of a graph. The negation of a question $\{a_q\}$ means checking being in the state described by the collection $\{\bar{a}_q\}$, $\bar{a}_q = 1 - a_q$.

Then consider a compound system of two identical objects which is prepared in such a way that if we put one of the questions X, Y, Z to the first object and the same question to the second one we always obtain exactly one YES and one NO. One can also put different questions to the objects. An answer to the question X to the first object is obtained, and we immediately know the answer to this question to the second object (namely, the opposite answer). Then one could ask about the validity of Bell's inequalities:

$$P(X_1 Y_1) + P(X_1 Z_1) \geq P(Y_1 Z_1)$$

However, no one of the questions X_1, Y_1, $X_1 Z_1$, and Y_1, Z_1 can be put directly. So to convert Bell's inequality to measurable form we equivalently have

$$P(X_1 \bar{Y}_2) + P(X_1 \bar{Z}_2) \geq P(Y_1 \bar{Z}_2) \tag{6.1}$$

where, for example, $P(X_1 \bar{Y}_2)$ is the probability to obtain YES for the question X to the first object and NO for the question Y to the second object.

Now let both objects be described by the graph $G_1(G_2)$ (Section 3). This graph simulates spin measurements on a spin-1/2 particle restricted to the XZ plane. Consider three elementary questions. Let X = "$S_x = 1/2$?," Y = "$S_a = 1/2$," and Z = "$S_z = 1/2$?," where a is an axis in the XZ plane forming the angle a with the z axis. These questions induce the following weights on the vertices 1, 2, 3, 4 on the graph G:

$$\begin{aligned}
X:&\quad x_1 = x_3 = 1/2, \qquad x_2 = 1,\ x_4 = 0 \\
Y:&\quad y_1 = (1 + \sin\alpha)/2, \qquad y_2 = (1 + \cos\alpha)/2 \\
&\quad y_1 y_3 = (1 - \sin\alpha)/2, \qquad y_4 = (1 - \cos\alpha)/2 \\
Z:&\quad z_1 = 1, \qquad z_2 = z_4 = 1/2, \qquad z_3 = 0
\end{aligned} \tag{6.2}$$

For the opposite questions we have $\bar{x}_1 = 1 - x_i$, and so on.

As was proposed, let our coupled system described by the graph G be in the state (d_{pq})—the eigenstate for the value -1 of the permutation operator. The state d is not an eigenstate for any question. The direct computation of scalar products $\langle q_{-}|ik\rangle$ yields

$$d_{pq}=\begin{vmatrix} 0 & 1/4 & 1/2 & 1/4 \\ 1/4 & 0 & 1/4 & 1/2 \\ 1/2 & 1/4 & 0 & 1/4 \\ 1/4 & 1/2 & 1/4 & 0 \end{vmatrix}$$

Now form the product questions occurring in (6.1). They are all factorizable and are calculated as pairwise products. For example, the collection of weights associated with the question $X_1 Y_2$ is $\{c_{ik}\}$, $C_{ik}=x_j(1-y_k)$, where the values of x_i and y_k are taken from (6.2).

The collection of weights associated with three questions from (3.1) are substituted into (4.1) to get the values of transition probabilities, which in our case are equal to

$$P(X_1, Y_2)=(1-\cos\alpha)/2$$
$$P(X_1Z_2)=1/2$$
$$P(Y_1Z_2)=(1+\sin\alpha)/2$$

If a is such that 1-cos a sin a, the inequality (3.1) is violated. We emphasize that the demonstrated violation of Bell's inequalities is essentially caused by the nondistributivity of the property lattice.

7. BOOLEIZATION THROUGH MEASUREMENT, THE ROLE OF CONSCIOUSNESS

One of the fundamental problems in quantum theory is the problem of measurement. London and Bauer (1939) discussed the idea that its solution is due to the special property of consciousness—its capacity for introspection. Introspection means the knowledge as an unambiguous identification of one's state of mind and it leads to wave packet reduction, so that probabilities appear. Here we develop this idea further. We connect introspection with the Boolean logic of the mind. Thus, if one considers the system: particle + apparatus + observer with mind, wave packet reduction appears. This is because the Boolean-minded observer (being part of a non-Boolean system) projects the whole onto his or her Boolean structure, which possesses the usual probability calculus. So it is the discrepancy between the non-Boolean strucure of the world and the Boolean nature of mind that leads to wave packet reduction.

The Booleization (projection of non-Boolean structure onto its Boolean substructure) is made by means of time. For instance, one can have the idea that both time and movement in time are "invented" by the Boolean mind in order to grasp the non-Boolean nature of the world as well as the body with which this mind is intimately connected.

To understand this, consider as an example the system of two spin-1/2 particles and their spin projections S_{z1} and S_{z2} on the z axis only. Construct the non-Boolean lattice of this system introducing the elementary questions corresponding to the following vectors in $H=H_1\otimes H_2$:

$$\begin{array}{ccc} \langle 1| = e_1e_1, & \langle 2| = e_1e_2 & \\ \langle 3| = (1/\sqrt{2})(e_1e_2 - e_2e_1), & \langle 4| = e_2e_1, & \langle 5| = e_2e_2 \end{array} \qquad (7.1)$$

Constructing the graph associated with the subspaces 1–5 in accordance with Section 1, we can see that the property lattice generated by this graph is isomorphic to the Boolean lattice 2^4 (generated by the graph of four disjoint vertices 1, 2, 4, 5). See Figure 1.

We need an ortholattice possessing the negative logic that we permanently apply (namely, the identification of a property by checking its opposite). In order to obtain the ortholattice, we shall take not the sublattice of $L(K)$, but the suborthoposet of the ortholattice $L(H)$ built in Section 4. The suborthoposet constructed, call it M, is generated by the considered properties $1, 2, \ldots, 5$ which are associated with the following elements of $L(H)$:

$$1 \mapsto \{11\}, \quad 2 \mapsto \{13\}, \quad 3 \mapsto \{q_-\}, \quad 4 \mapsto \{31\}, \quad 5 \mapsto \{33\}$$

The lattice M is constructed as the lattice 2^4 generated by elements 1, 2, 4, 5 with two additional elements 3 and $3^\perp$, which are connected with other elements as shown in Figure 2. Due to the presence of these additional elements, M is non-Boolean, and thus one cannot define a usual probability measure on it; instead, we introduce some weights. For example, for a singlet state one can have the following collection of weights $\{w_i\}$:

$$W_1 = W_5 = 0, \qquad W_2 = W_4 = 1/2, \qquad W_3 = 1$$

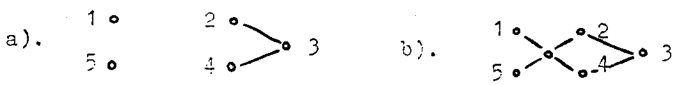


Fig. 1. (a) The graph associated with the collection (7.1). (b) The graph generating the ortholattice M.

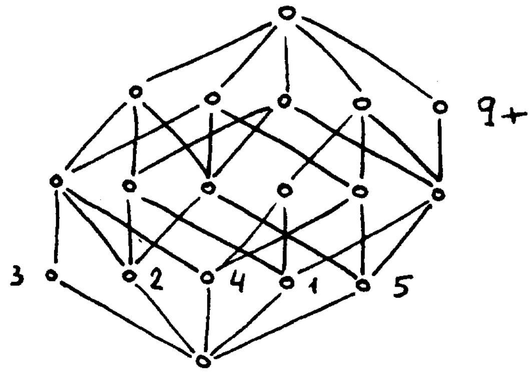


Fig. 2

Due to nondistributivity

$$2 = 2 \wedge (4 \vee 3) \neq (2 \wedge 4) \vee (2 \wedge 3) = 0 \vee 0 = 0$$

The property $1 \vee 2$ corresponds to the observation of $S_z^{(1)} = +1/2$ without the observation of anything for a second particle. The occurrence of $S_z^{(1)} = +1/2$ does not mean that 2 occurs, because w_i are not probabilities, and thus 2 cannot be called an event in the described experimental situation.

It is only if one "neglects" the element 3 that one obtains probabilities corresponding to a Boolean lattice. In order to "neglect" 3, the observer considers some other moment of time, so that 3 is now in the past and only 1, 2, 4, and 5 are actual at the present moment. This corresponds to the usual preparation and measurement procedures in quantum mechanics. Formally this can mean that we have a Hilbert space with a superselection rule associated with time. One can say that there are two Hilbert spaces parametrized by time moments t_1 and t_2, so that performing the measurements commutes with the permutation operator at moment t_1 and with the local operators $S_z(1)$ and $S_z(2)$ at moment t_2. The Boolean observer prepares the system at moment t_1 and obtains with probabilities $1/2$ these or those values of $S_z(1)$, $S_z(2)$ at moment t_2.

This corresponds to the well-known problem (D'Espagnat, 1976) in the EPR experiments. Let the two-particle system be prepared in the singlet state. One of the two observers observes a spin S equal to $+1/2$ for one particle and with probability equal to 1^z (as he calls it) says that the other particle now has $S_z = -1/2$. But this statement for the other particle is not an event in Minkowski space-time, but only an "objectively existing potentiality" (Fock, 1965). But if the other also looks at the other particle and checks whether $S_z = -1/2$ (indeed it cannot be anything else, since this is $2 \vee 5$), then one concludes that the state of the two-particle system is $\{13\}$.

This conclusion does not depend on weights, but is just the consequence of the structure of the lattice [see also Bitbol (1983) for the many-world interpretation]. So, it is for Boolean observers that events appear.

One can go still further and take an ensemble of copies of elements which are enumerated by time t:

This ensemble of copies of the same quantum system is often used as a claim for the Everett–Wheeler–DeWitt many-world interpretation (Everett, 1957). However, here we interpret it in the other way, as a method of "Booleization" of non-Boolean logics.

It is well known that $|\psi\rangle_\infty$ is an eigenstate for different "frequency operators." These frequency operators are in one-to-one correspondence with all noncommuting observables of the system, for example, $\hat{A}$, $\hat{B}$ such that $[\hat{A}, \hat{B}] \neq 0$ are associated with frequency operators

$$\hat{f}^k_A = \lim_{n\to\infty} \sum i_1 \cdots i_n |i_1, 1\rangle \cdots |i_n, N\rangle (1/N) \sum_{\alpha=1}^{N} \delta_{ki_\alpha}$$

$$\times \langle i_n, N| \cdots \langle i_1, 1|$$

$$\hat{f}^k_B = \lim_{n\to\infty} \sum j_1 \cdots j_n |j_1, 1\rangle \cdots |j_n, N\rangle (1/N) \sum_{\alpha=1}^{N} \delta_{kj_\alpha}$$

$$\times \langle j_n, N| \cdots \langle j_1, 1|$$

where $|i_k\rangle$, $|j_k\rangle$ are the eigenstates of $\hat{A}$, $\hat{B}$, respectively.

Due to the $1/N$ factor, the operators $\hat{f}^k_A$ and $\hat{f}^k_B$ do commute, $\hat{A}$, $\hat{B}$ do not. So, a nondistributive lattice containing $\hat{A}$, $\hat{B}$ corresponds to a distributive lattice where instead of $\hat{A}$, $\hat{B}$ we have appropriate frequencies. It is also important that the "Booleization" is possible when $N \to \infty$, which corresponds to the continuous limit, so time must be continuous: Macroscopic observers observe these frequencies as macroscopic events.

8. SUMMARY

Two-particle quantum systems with spin can be simulated by classical automata described by graphs. These graphs are associated with nondistributive property lattices of these quantum systems. We emphasize that to nonlocal properties of a quantum system being in a certain eigenstate of the permutation operator there correspond merely some additional vertices in the graph which have nothing "nonlocal" in their nature. This leads to the possibility of violating Bell's inequalities in classical systems described by graphs (see Section 6) without violating relativity theory.

The subjective interpretation of quantum mechanics of von Neumann, London, and Bauer can be connected with the Boolean nature of mind grasping the non-Boolean nature of the world, which results in the projection postulate: wave packet reduction. A simple example is given for a two-particle system with spin.

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